

Mark scheme

Practice paper 1

$$1 \quad \mathbf{n}_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}$$

(A1) (A1) (M1)

$$= \frac{2+2-1}{\sqrt{6} \cdot \sqrt{6}} = \frac{1}{2} \Rightarrow \theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

(A1) (A1) [5 marks]

$$2 \quad \mathbf{a} \quad f(-x) = (-x)\sin^3(-x) = (-x)(-\sin^3 x) = x\sin^3 x = f(x)$$

(M1) (A1) (A1) even

$$\mathbf{b} \quad g(-x) = \sqrt[3]{\frac{e^{(-x)^2}+1}{-x}} = \sqrt[3]{\frac{e^{x^2}+1}{-x}} = -\sqrt[3]{\frac{e^{x^2}+1}{x}} = -g(x)$$

(M1) (A1) (A1) odd [6 marks]

$$3 \quad \mathbf{a} \quad \lim_{x \rightarrow 2} \frac{2x^2 - 5x + 2}{3x^2 - 4x - 4} = \lim_{x \rightarrow 2} \frac{(2x-1)(x-2)}{(3x+2)(x-2)}$$

$$= \lim_{x \rightarrow 2} \frac{2x-1}{3x+2} = \frac{2 \cdot 2 - 1}{3 \cdot 2 + 2} = \frac{3}{8}$$

(M1) (A1)

(A1)

$$\mathbf{b} \quad \lim_{x \rightarrow 0} \frac{\sin 3x}{2x} = \lim_{x \rightarrow 0} \frac{3}{2} \cdot \frac{\sin 3x}{3x}$$

(M1) (A1)

$$= \frac{3}{2} \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x} = \frac{3}{2} \cdot 1 = \frac{3}{2}$$

(A1) [6 marks]

$$4 \quad \mathbf{a} \quad (a+2d)^2 = a(a+5d)$$

(M1) (A1)

$$a^2 + 4ad + 4d^2 = a^2 + 5ad \Rightarrow ad = 4d^2 \Rightarrow a = 4d$$

(A1) (A1)

$$\mathbf{b} \quad (a+4d)^2 = a(a+12d) \Rightarrow (8d)^2 = 4d(16d)$$

(M1) (A1)

$$\Rightarrow 64d^2 = 64d^2$$

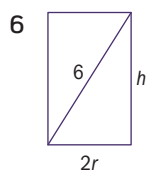
(A1) [7 marks]

$$5 \quad \mathbf{a} \quad \frac{1}{2}(8-1)! = 2520$$

(M1) (A1)

$$\mathbf{b} \quad P = \frac{\frac{1}{2} \cdot 2 \cdot (7-1)!}{\frac{1}{2}(8-1)!} = \frac{2 \cdot 6!}{7!} = \frac{2}{7}$$

(M1) (M1) (A1) [5 marks]



$$h^2 + 4r^2 = 36 \Rightarrow r^2 = 9 - \frac{1}{4}h^2$$

(A1)

$$V = r^2 \pi h = \left(9 - \frac{1}{4}h^2\right) \pi h = \left(9h - \frac{1}{4}h^3\right) \pi$$

(M1) (A1)

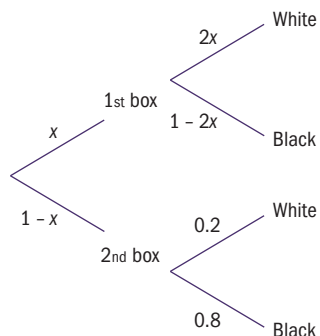
$$\frac{dV}{dh} = \left(9 - \frac{3}{4}h^2\right) \pi = 0 \Rightarrow h = \sqrt{12}$$

(M1) (A1)

$$r = \sqrt{9 - \frac{1}{4} \cdot 12} = \sqrt{6} \text{ cm}$$

(A1) [6 marks]

7 a



b $P(W) = x \cdot 2x + (1-x) \cdot 0.2 = 0.44$

$$2x^2 - 0.2x - 0.24 = 0 \Rightarrow (2x + 0.6)(x - 0.4) = 0$$

$$\cancel{x_1 = -0.3}, x_2 = 0.4$$

(M1) (A1)

(M1) (A1)

(A1) [7 marks]

8 $2 \times 25^x = 3 \times 10^x + 5 \times 4^x \Rightarrow 2 \times \frac{25^x}{4^x} = 3 \times \frac{10^x}{4^x} + 5 \times \frac{4^x}{4^x}$

$$\Rightarrow 2 \times \left(\frac{5}{2}\right)^{2x} - 3 \times \left(\frac{5}{2}\right)^x - 5 = 0$$

(M1)

(A1)

Let's use a substitution $\left(\frac{5}{2}\right)^x = t \Rightarrow 2t^2 - 3t - 5 = 0$

(M1)

$$(2t-5)(t+1) = 0 \Rightarrow t_1 = \frac{5}{2}, \cancel{t_2 = -1}$$

(A1) (A1)

$$\left(\frac{5}{2}\right)^x = \frac{5}{2} \Rightarrow x = 1$$

(A1) [6 marks]

9 a $\log_x 10 = \log_x 5 + \log_x 2$

(M1)

$$= \frac{1}{\log_5 x} + \frac{1}{\log_2 x} = \frac{1}{p} + \frac{1}{q}$$

(A1) (A1)

b $\log x = \frac{1}{\log_x 10}$

(M1)

$$= \frac{1}{\frac{1}{p} + \frac{1}{q}} = \frac{pq}{p+q}$$

(A1) (A1)

[6 marks]

10 Let's assume that $\cos \alpha \neq 0$ and $\sin \alpha \neq 0$.

$$\begin{cases} \sin \alpha \cdot x - \cos \alpha \cdot y = 1 \\ \cos \alpha \cdot x + \sin \alpha \cdot y = 1 \end{cases} \Rightarrow \begin{cases} \sin^2 \alpha \cdot x - \sin \alpha \cdot \cos \alpha \cdot y = \sin \alpha \\ \cos^2 \alpha \cdot x + \cos \alpha \cdot \sin \alpha \cdot y = \cos \alpha \end{cases} +$$

(M1)

$$\underbrace{(\sin^2 \alpha + \cos^2 \alpha)}_1 x = \sin \alpha + \cos \alpha \Rightarrow x = \sin \alpha + \cos \alpha$$

(A1)

To find y we use the second equation

$$\cos \alpha (\sin \alpha + \cos \alpha) + \sin \alpha \cdot y = 1$$

$$\Rightarrow \sin \alpha \cdot y = 1 - \cos^2 \alpha - \sin \alpha \cdot \cos \alpha$$

(M1)

$$\Rightarrow \sin \alpha \cdot y = \sin^2 \alpha - \sin \alpha \cdot \cos \alpha, \quad \sin \alpha \neq 0$$

(A1)

$$\Rightarrow y = \sin \alpha - \cos \alpha$$

(A1)

If $\begin{cases} \sin \alpha = 0 \Rightarrow \cos \alpha \neq 0 \Rightarrow y = -\sec \alpha, x = \sec \alpha \\ \cos \alpha = 0 \Rightarrow \sin \alpha \neq 0 \Rightarrow y = \csc \alpha, x = \csc \alpha \end{cases}$

so we always have a unique solution.

(R1) (R1) [6 marks]

$$11 \text{ a } x^2 - x^3 \neq 0 \Rightarrow x^2(1-x) \neq 0 \quad (\text{M1})$$

$$\text{b } D(f) = \{x \in \mathbb{R} : x \neq 0, 1\} \quad (\text{A1})$$

$$= \lim_{x \rightarrow \pm\infty} \frac{1 - \frac{2}{x} + \frac{5}{x^3}}{\frac{1}{x} - 1} = -1 \quad (\text{M1}) \quad (\text{A1}) \quad (\text{A1})$$

$$\text{c } \frac{x^3 - 2x^2 + 5}{x^2 - x^3} = 1 \quad (\text{M1})$$

$$x^3 - 2x^2 + 5 = x^2 - x^3 \Rightarrow 2x^3 - 3x^2 + 5 = 0 \quad (\text{A1})$$

On inspection, by using synthetic division we obtain possible x -values.

$$\begin{array}{r|rrrr} & 2 & -3 & 0 & 5 \\ \cdot(-1) & & + & + & + \\ & -2 & 5 & -5 & \\ \hline & 2 & -5 & 5 & 0 \end{array}$$

$$2x^3 - 3x^2 + 5 = 0 \Rightarrow (x+1)(2x^2 - 5x + 5) = 0 \quad (\text{M1}) \quad (\text{A1})$$

$$x_1 = -1, x_{2,3} \notin \mathbb{R} \Rightarrow (-1, 1) \quad (\text{A1}) \quad [10 \text{ marks}]$$

$$12 \text{ a } \int \sin^n(x) dx \int \sin^{n-1}(x) \cdot \sin x dx \quad (\text{M1})$$

$$\text{Let } \begin{cases} \sin^{n-1}(x) = u \Rightarrow (n-1)\sin^{n-2}(x) \cdot \cos x dx = du \\ \sin x dx = dv \Rightarrow -\cos x = v \end{cases} \quad (\text{A1}) \quad (\text{A1})$$

$$= \sin^{n-1}(x) \cdot (-\cos x) - \int (-\cos x) \cdot (n-1) \sin^{n-2}(x) \cdot -\cos x dx \quad (\text{A1})$$

$$= -\sin^{n-1}(x) \cos x + (n-1) \int \sin^{n-2}(x) \cdot \cos^2 x dx$$

$$= -\sin^{n-1}(x) \cos x + (n-1) \int \sin^{n-2}(x) \cdot (1 - \sin^2 x) dx \quad (\text{A1})$$

$$= -\sin^{n-1}(x) \cos x + (n-1) \int \sin^{n-2}(x) dx - (n-1) \int \sin^n(x) dx \quad (\text{A1})$$

$$n \int \sin^n(x) dx = -\sin^{n-1}(x) \cos x + (n-1) \int \sin^{n-2}(x) dx$$

$$\int \sin^n(x) dx = -\frac{1}{n} \sin^{n-1}(x) \cos x + \frac{n-1}{n} \int \sin^{n-2}(x) dx \quad (\text{A1})$$

$$\int_0^{\frac{\pi}{2}} \sin^n(x) dx = \left[-\frac{1}{n} \sin^{n-1}(x) \cos x \right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2}(x) dx \quad (\text{A1})$$

$$= -\frac{1}{n} \left(\left(\sin^{n-1}\left(\frac{\pi}{2}\right) \cos \frac{\pi}{2} \right) - \left(\underbrace{\sin^{n-1}(0)}_0 \cos 0 \right) \right) + \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2}(x) dx \quad (\text{A1})$$

$$= \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2}(x) dx \quad (\text{AG})$$

$$\text{b } \int_0^{\frac{\pi}{2}} \sin^4(x) dx = \frac{3}{4} \int_0^{\frac{\pi}{2}} \sin^2(x) dx \quad (\text{A1})$$

$$= \frac{3}{4} \left(\frac{1}{2} \int_0^{\frac{\pi}{2}} dx \right) \quad (\text{A1})$$

$$= \frac{3}{8} \left(\frac{\pi}{2} - 0 \right) = \frac{3\pi}{16} \quad (\text{A1}) \quad [12 \text{ marks}]$$

$$13 \text{ a } \sum_{r=1}^n (3r-1) = 2 + 5 + 8 + \dots + (3n-1)$$

$$n=1 \Rightarrow 2 = \frac{1}{2} \cdot 1 \cdot (3 \cdot 1 + 1) \quad (\text{M1})$$

$$2 = 2, \text{ true for } n = 1 \quad (\text{A1})$$

Assume that the formula works for $n = k$.

$$\sum_{r=1}^k (3r-1) = \frac{1}{2} k(3k+1) \quad (\text{M1})$$

$$n = k + 1 \Rightarrow \sum_{r=1}^{k+1} (3r-1) = \underbrace{2 + 5 + 8 + \dots + (3k-1)}_{\text{Assumption}} + (3k+2) \quad (\text{M1})$$

$$= \frac{1}{2} k(3k+1) + (3k+2) = \frac{1}{2} (3k^2 + k + 6k + 4) \quad (\text{A1})$$

$$= \frac{1}{2} (3k^2 + 7k + 4) = \frac{1}{2} (k+1)(3k+4) \quad (\text{A1}) \quad (\text{A1})$$

The formula works for $n = 1$ and, from the assumption that it works for $n = k$, we obtained that it works for $n = k + 1$, therefore it works for all $n \in \mathbb{Z}^+$.

(R1)

$$\text{b } 1 - 2 + 3 + 4 - 5 + \dots + (3n-2) - (3n-1) + 3n \quad (\text{M1}) \quad (\text{A2})$$

$$= \frac{3n \cdot (3n+1)}{2} - 2 \cdot \frac{1}{2} n(3n+1) \quad (\text{A1}) \quad (\text{A1})$$

$$= \frac{n \cdot (3n+1)}{2} \quad (\text{A1}) \quad [14 \text{ marks}]$$

$$14 \text{ a } z^2 + z + 1 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \quad (\text{M1}) \quad (\text{A1})$$

$$\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}, \omega^* = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \quad (\text{A1}) \quad (\text{A1})$$

$$\text{b i } \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)^2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = \omega^* \quad (\text{A1})$$

$$\text{ii } \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right)^2 = \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \omega \quad (\text{A1})$$

$$\text{iii } \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)^3 = \cos \frac{6\pi}{3} + i \sin \frac{6\pi}{3} = \cos 2\pi + i \sin 2\pi = 1 \quad (\text{A1})$$

$$\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right)^3 = \cos \frac{12\pi}{3} + i \sin \frac{12\pi}{3} = \cos 4\pi + i \sin 4\pi = 1 \quad (\text{A1})$$

$$\text{c i } (1+\omega)^{3n} = 1 + \binom{3n}{1}\omega + \binom{3n}{2}\omega^2 + \binom{3n}{3}\omega^3 + \dots \quad (\text{M1})$$

$$= 1 + \binom{3n}{1}\omega + \binom{3n}{2}\omega^* + \binom{3n}{3} + \dots \quad (\text{A1})$$

$$\text{ii } (1+\omega^*)^{3n} = 1 + \binom{3n}{1}\omega^* + \binom{3n}{2}(\omega^*)^2 + \binom{3n}{3}(\omega^*)^3 + \dots \quad (\text{M1})$$

$$= 1 + \binom{3n}{1}\omega^* + \binom{3n}{2}\omega + \binom{3n}{3} + \dots \quad (\text{A1})$$

$$\text{d i } 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \quad (\text{M1}) \quad (\text{A1})$$

$$\text{ii } 1 + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \quad (\text{A1})$$

$$\text{e} \quad \binom{3n}{0} + \binom{3n}{1} + \binom{3n}{2} + \dots + \binom{3n}{3n} = (1+1)^{3n} = 2^{3n} \quad (\text{M1}) \quad (\text{A1})$$

$$\text{f} \quad \left. \begin{aligned} 2^{3n} &= \binom{3n}{0} + \binom{3n}{1} + \binom{3n}{2} + \binom{3n}{3} + \dots + \binom{3n}{3n} \\ (1+\omega)^{3n} &= \binom{3n}{0} + \binom{3n}{1}\omega + \binom{3n}{2}\omega^* + \binom{3n}{3} + \dots + \binom{3n}{3n} \\ (1+\omega^*)^{3n} &= \binom{3n}{0} + \binom{3n}{1}\omega^* + \binom{3n}{2}\omega + \binom{3n}{3} + \dots + \binom{3n}{3n} \end{aligned} \right\} + \quad (\text{M1}) \quad (\text{A1})$$

$$\begin{aligned} & 2^{3n} + \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^{3n} + \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right)^{3n} \\ &= 3 \cdot \left(\binom{3n}{0} + \binom{3n}{3} + \binom{3n}{6} + \dots + \binom{3n}{3n} \right) \end{aligned} \quad (\text{A1}) \quad (\text{A1})$$

$$\begin{aligned} & 2^{3n} + \left(\cos n\pi + i \underbrace{\sin n\pi}_0 \right) + \left(\cos 5n\pi + i \underbrace{\sin 5n\pi}_0 \right) \\ &= 3 \cdot \left(\binom{3n}{0} + \binom{3n}{3} + \binom{3n}{6} + \dots + \binom{3n}{3n} \right) \end{aligned} \quad (\text{A1}) \quad (\text{A1})$$

$$\text{Since } \cos(n\pi) = \cos(5n\pi) = \begin{cases} 1, n \text{ is even} \\ -1, n \text{ is odd} \end{cases} \text{ we can write} \quad (\text{R1})$$

$$\binom{3n}{0} + \binom{3n}{3} + \binom{3n}{6} + \dots + \binom{3n}{3n} = \frac{2^{3n} + 2 \cos(n\pi)}{3}. \quad (\text{AG}) \quad [24 \text{ marks}]$$